§ Linear (in)dependence.

Def: V vector space over F. A subset $S \subset V$ is said to be **linearly dependent** if $\exists$ distinct $\vec{u_1}, \cdots, \vec{u_n} \in S$ and non-zero scalars $a_1, \cdots, a_n \in F \setminus \{0\}$ s.t.

$$\boxed{a_1\vec{u_1} + \cdots + a_n\vec{u_n} = \vec{0}}$$

Otherwise, it is said to be **linearly independent**.

Rmk:

- The empty set $\phi \subset V$ is linearly independent (convention)
- If $\vec{0} \in S$, then S is linearly dependent. $(a \cdot \vec{0} = \vec{0})$
- If $S = \{\vec{u}\}$ and $\vec{u} \neq \vec{0}$. then S is linearly independent.

$\left(a\vec{u} = \vec{0}.\ \text{If } a \neq 0,\ \underbrace{\cancel{a^{-1}a}}_{1} \cdot \vec{u} = a^{-1} \cdot \vec{0} = \vec{0} \right).$

Equivalent Definitions:

(1) S is linear independent.

(2). Each $\vec{x} \in \mathrm{Span}(S)$ can be expressed in a unique way as a linear combination of vectors of S.

(3). If $a_1\vec{u_1} + \cdots + a_n\vec{u_n} = \vec{0}$ for $\vec{u_1}, \cdots \vec{u_n} \in S$ and $a_1, \cdots, a_n \in F$ then we must have $a_1 = \cdots = a_n = 0$.

pf is straightforward: (1) $\Leftrightarrow$ (3): def ; (2) $\Rightarrow$ (3): take $\vec{x} = \vec{0}$.

(3) $\Rightarrow$ (2): If $\vec{x} = a_1\vec{u_1} + \cdots + a_n\vec{u_n} = b_1\vec{v_1} + \cdots + b_m\vec{v_m}$

$\Rightarrow \vec{0} = a_1\vec{u_1} + \cdots + a_n\vec{u_n} - b_1\vec{v_1} - \cdots - b_m\vec{v_m}$

$\square$

Example: For $k = 0, \cdots, n$. Let $f_k(x) = 1 + x + \cdots + x^k$,

Then $S := \{f_0(x), \cdots, f_n(x)\} \subset P_n(F)$ is a linearly indep. subset.

pf:

$$
\begin{aligned}
0 &= a_0 f_0(x) + \cdots + a_n f_n(x) \\
&= a_0 \cdot 1 + a_1(1+x) + \cdots + a_n(1 + x + \cdots + x^n) \\
&= \underbrace{(a_0 + a_1 + \cdots + a_n)}_{=0} + \underbrace{(a_1 + \cdots + a_n)}_{=0} x + \cdots + \underbrace{a_n}_{=0} x^n .
\end{aligned}
$$

$$
\Rightarrow \begin{cases} a_0 + a_1 + \cdots + a_n = 0 \\ a_1 + \cdots + a_n = 0, \\ \quad \ddots \\ a_{n-1} + a_n = 0 \\ a_n = 0 \end{cases} \Rightarrow a_0 = \cdots = a_n = 0,
$$

$\square$

Proposition 1: Let S be a linearly indep subset of V.
Let $\vec{v} \in V \setminus S$. Then $S \cup \{\vec{v}\}$ is linearly dependent
iff. $\vec{v} \in \mathrm{Span}(S)$.

pf: $(\Rightarrow)$: Suppose $S \cup \{\vec{v}\}$ is linearly dependent.

Then, $a_1 \underbrace{\overrightarrow{u_1}}_{=\vec{v}} + \cdots + a_n \overrightarrow{u_n} = 0$
for some $\overrightarrow{u_1}, \cdots, \overrightarrow{u_n} \in S \cup \{\vec{v}\}$ and $a_1, \cdots a_n \in F \setminus \{0\}$

Since S is lin. indep, one of $\overrightarrow{u_j}$'s, say $\overrightarrow{u_1}$, is $\vec{V}$.

Hence $\vec{v} = a_1^{-1}(-a_2\overrightarrow{u_2} - \cdots - a_n\overrightarrow{u_n}) \in \mathrm{Span}(S)$.

$(\Leftarrow)$: If $\vec{v} \in \operatorname{Span}(S)$, then we can write $\vec{v} = b_1\vec{v_1} + \cdots + b_m\vec{v_m}$ for some $\vec{v_1}, \cdots, \vec{v_m} \in S$ and $b_1, \cdots, b_m \in F$.

$\Rightarrow \quad 0 = -\vec{v} + b_1\vec{v_1} + \cdots + b_m\vec{v_m}$ is a non-trivial linear relation

So $S \cup \{v\}$ is linearly dependent. 17

Proposition 2: Let S be a linearly dependent subset of V. Then $\exists \vec{v} \in S$ s.t. $\mathrm{span}(S) = \mathrm{span}(S - \{\vec{v}\})$

pf: S lin. dep $\Rightarrow$ $\vec{0} = b_1 \underbrace{\vec{v_1}}_{=:\vec{v}} + \cdots + b_m \vec{v_m}$,

$\vec{v_1}, \cdots, \vec{v_m} \in S$, $b_1, \cdots, b_m \in F \setminus \{0\}$

$\Rightarrow \vec{v_1} = -b_1^{-1}(b_2\vec{v_2} + \cdots + b_m\vec{v_m})$

For $\vec{w} \in \mathrm{span}\, S$, $\vec{W} = a_1 \vec{v_1} + \cdots + a_n \vec{v_n}$

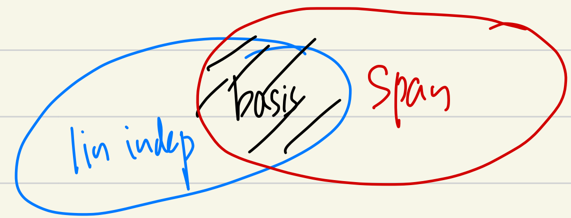
$= -a_1 b_1^{-1}(b_2\vec{v_2} + \cdots + b_m\vec{v_m}) + a_2\vec{v_2} + \cdots + a_n\vec{v_n} = \mathrm{Span}(S - \{\vec{v_1}\})$

$\Rightarrow \mathrm{Span}(S) = \mathrm{span}(S - \{\vec{v_1}\})$ □

§ Basis

Def: A basis for a vector space V is a subset $\beta \subset V$ s.t.

- β is linearly indep.
- β spans V.



Prop: Let V be a vector space and $\beta = \{\vec{u_1}, \cdots, \vec{u_n}\} \subset V$ a subset.
(Alt. Definition)
Then β is a basis for V iff $\forall \vec{v} \in V$, $\exists!\, a_1, \cdots, a_n \in F$
there exist unique
s.t. $\vec{v} = a_1\vec{u_1} + \cdots + a_n\vec{u_n}$.

Example. • $\{\vec{e_1}, \dots, \vec{e_n}\}$ is a basis for F^n. Called the "standard basis".

• $\{1, x, \dots, x^n\}$ is a standard basis for $P_n(F)$.

$\{f_0(x)=1,\ f_1(x)=1+x,\ \dots,\ f_n(x)=1+x+\dots+x^n\}$ is also a basis

$\{1, x, x^2, \dots\}$ is a basis for $P(F)$

• $\{E_{ij} : 1\le i\le m,\ 1\le j\le n\}$, where $E_{ij} = \begin{array}{cc} j^{th} & \\ \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} & i^{th} \end{array}$ is a basis for $M_{m\times n}(F)$

- Span v.s lin. indep. v.s. basis

"big enough" (Span) "small enough" (lin. indep.)

Theorem 1: Suppose S is a finite spanning set for V. Then $\exists\, \beta \subset S$ which is a basis for V.

(So a finite spanning set can be reduced to a basis)

Pf: If S is linearly indep. then we can take $\beta = S$.

Otherwise, by an earlier result, $\exists \vec{v_1} \in S$ s.t. $\underbrace{\text{span}(S \setminus \{\vec{v_1}\})}_{=: S_1} = \text{span}(S)$

If $S_1 := S \setminus \{\vec{v_1}\}$ is linearly indep, then take $\beta = S_1$

Otherwise, $\exists \vec{v_2} \in S$ s.t. $\underbrace{\text{span}(S_1 \setminus \{\vec{v_2}\})}_{=: S_2} = \text{span}(S_1)$.

Repeat this process ...

$S \rightsquigarrow S - \{\vec{v_1}\} \rightsquigarrow S - \{\vec{v_1}, \vec{v_2}\} \rightsquigarrow \cdots$

Same span.

Since S is assumed to be finite,

we will arrive at a linearly indep subset S_k.

s.t. $\text{span}(S_k) = \text{span}(S) = V$

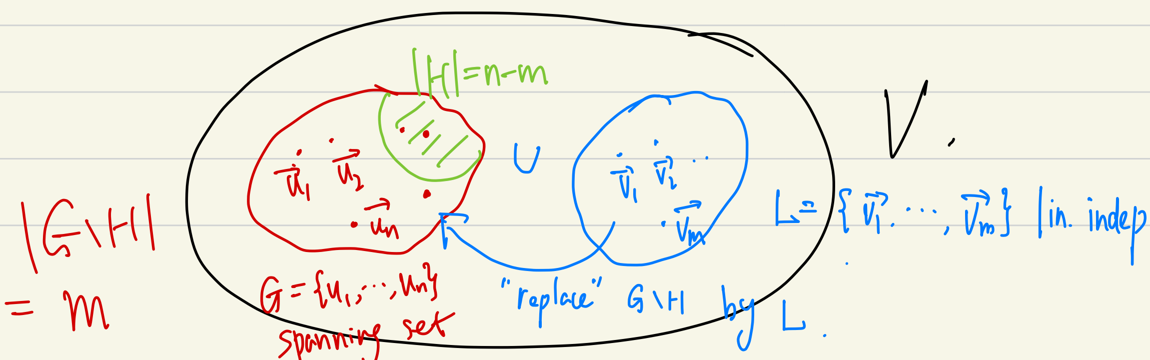
$\Rightarrow \beta = S_k$ is a basis. $\square$

Theorem 2. (Replacement thm) Let $L \subset V$ be a linearly indep. subset consisting of m vectors.
Let $G \subset V$ be a spanning set consisting of n vectors.

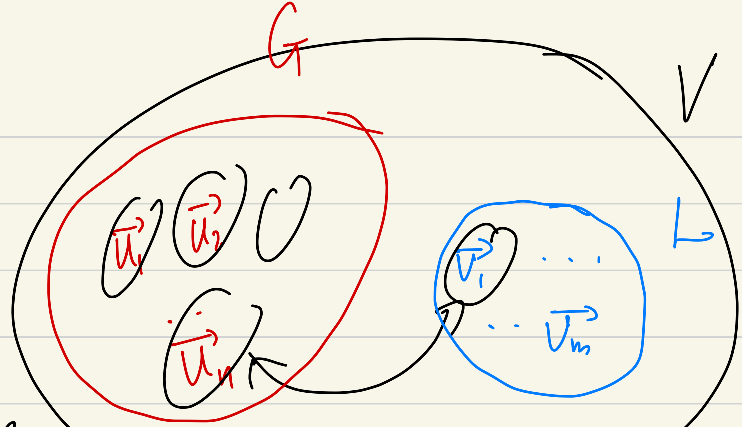
Then: $m \le n$. ($|\text{lin. ind. set}| \le |\text{spanning set}|$)

- $\exists\, H \subset G$ consisting of exactly $(n-m)$ vectors, s.t. $L \cup H$ spans V. (Alternatively, replace $G \setminus H$ by L)

(A lin indep set can be extended to a spanning set)



Idea:



$\vec{v}_1 = c_1\vec{u}_1 + \cdots + c_k\vec{u}_k$.

replacement

$\operatorname{Span}(G \setminus \{u_1\} \cup \{v_1\}) = \operatorname{span} G$.

Pf by induction on $m \geq 0$:

For $m=0$. $L=\emptyset$. Simply take $H=G$.

Suppose the statement is true for $m \geq 0$, need to prove for $m+1$.

Let $L=\{\vec{v}_1, \cdots, \vec{v}_{m+1}\}$ lin. indep subset of V. of size $m+1$.

Let $L'=\{\vec{v}_1, \cdots, \vec{v}_m\} \subset L$ lin indep of size m.

By the induction hypothesis, we have $m \leq n$. and $\exists\, H'=\{\vec{u}_1, \cdots \vec{u}_{n-m}\} \subset G$

s.t. $L' \cup H' = \{\vec{v}_1, \cdots, \vec{v}_m, \vec{u}_1, \cdots \vec{u}_{n-m}\}$ spans V.

Hence, $\exists\, a_1, \cdots, a_m, b_1, \cdots, b_{n-m} \in F$ s.t.

$$\vec{v}_{m+1} = a_1\vec{v}_1 + \cdots + a_m\vec{v}_m + \underset{\neq}{b_1}\vec{u}_1 + \cdots + b_{n-m}\vec{u}_{n-m}.$$

But $L = \{\vec{v}_1, \cdots, \vec{v}_{m+1}\}$ is lin. indep,

so $n-m \geq 1 \iff n \geq m+1$

and one of b_k's, say b_1, is nonzero.

$$\Rightarrow \vec{u}_1 \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_m, \vec{v}_{m+1}, \vec{u}_2, \cdots, \vec{u}_{n-m}\}$$

Hence, if we take $H := \{\vec{u}_2, \cdots, u_{n-m}\}$, then $L \cup H$ spans V.

This completes the induction argument. $\square$.

§ Dimension.

Theorem. Let V be a vector space having a finite basis. Then every basis of V contains the same # of vectors.

pf: Let β and γ be two basis for V.

β spans V, γ lin. indep

$\Rightarrow$ $|\gamma| \le |\beta|$ by Replacement Thm.

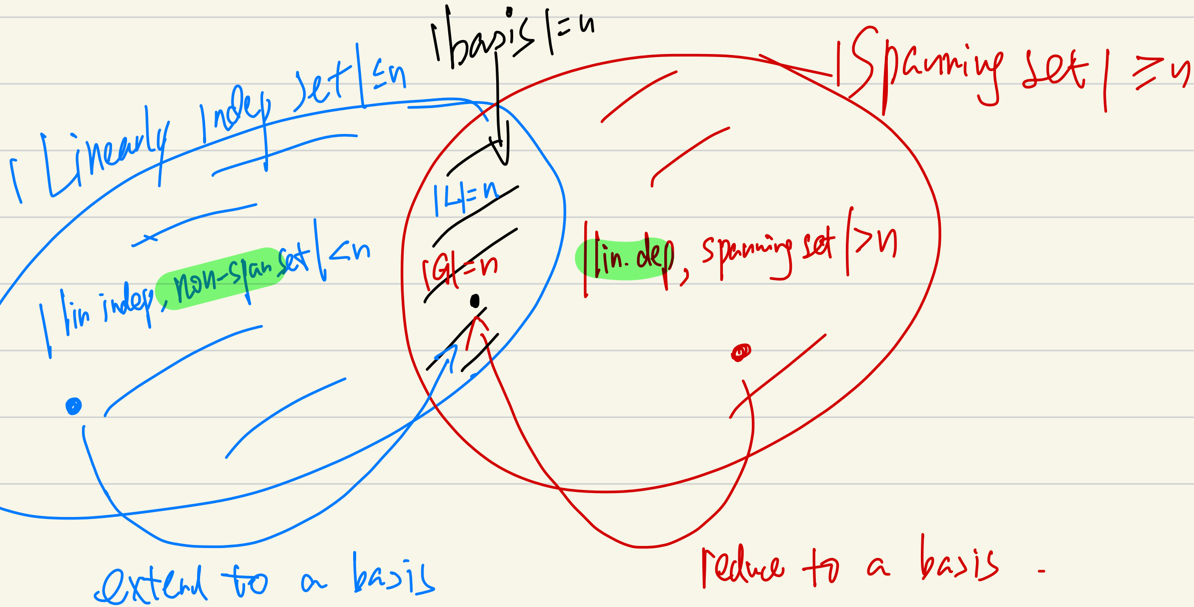
Similarly, γ spans V, β lin indep

$\Rightarrow$ $|\beta| \le |\gamma|$

Hence, $|\beta| = |\gamma|$. $\square$

Def:
- A vector space V is called finite-dimensional if it has a finite basis. The dimension of V, denoted as $\dim(V)$, is the # of vectors in a basis for V.
- A vector space that is not finite-dimensional is called infinite-dimensional.

Theorem



More precisely, Suppose V is an n-dimensional vector space.

- Any finite spanning set has $\geq n$ vectors,
  and a spanning set with exactly n vectors is a basis.
- Any lin. indep set has $\leq n$ vectors.
  -- -- -- -- -- exactly n vectors is a basis.
- Every spanning set (of size $>n$) can be reduced to a basis.
- -- lin indep .. ... $<n$ ... extended to a basis.

Pf: Let β be a basis ; L lin indep ; G spanning set.

- View β lin indep. Replacement Thm $\Rightarrow$ $n = |\beta| \le |G|$
  - - - - spanning - - - - . . $|L| \le |\beta| = n$

- Suppose $\#G > n$ $\Rightarrow$ lin dependent
  $\Rightarrow$ can reduce G to a basis.

- Suppose $\#G = n$. If it's lin. dep.
  $\Rightarrow$ can further reduce to a basis of size $< n$. Contradiction!

  Hence, G must be lin. indep.
  $\Rightarrow$ G is a basis.

basis, thus span.

- Suppose $|L| = m < n$

  By the Replacement Thm, Can find $H \subset \beta$ of size $(n-m)$

  s.t. $L \cup H$ spans V.

  $|L \cup H| = n = \dim V. \quad \Rightarrow L \cup H$ is a basis.

- Suppose $|L| = n$.

  Replacement Thm

  $\Rightarrow$ Can find H of size $n - n = 0$.

  i.e., $H = \phi$.

  s.t. $L \cup H = L$ spans V. $\Rightarrow L$ is a basis. $\square$

Example: $P_n(F)$ has a basis $\{1, x, \cdots, x^n\}$

$\dim = n+1$

$$\{f_0(x)=1,\ f_1(x)=1+x,\ \cdots,\ f_n(x)=1+x+\cdots+x^n\}$$

is lin indep of size $n+1 = \dim P_n(F)$

$\Rightarrow$ it is a basis.

Theorem: Let V be a finite-dimensional vector space.
Then any subspace $W \subset V$ is finite dim and $\dim(W) \leq \dim(V)$.
Moreover, if $\dim W = \dim V$, then $W = V$.

pf: Let $n = \dim(V)$. If $W = \{0\}$, obvious.
Otherwise, W contains a non-zero vector $\vec{u_1}$, so $\{\vec{u_1}\}$ is lin. indep.
If $\operatorname{span}\{\vec{u_1}\} = W$, then $\{\vec{u_1}\}$ is a basis of W.

Otherwise, choose $\vec{u_2} \in W \setminus \operatorname{span}\{u_1\}$ $\Rightarrow$ $\{\vec{u_1}, \vec{u_2}\}$ is lin. indep.
If $\operatorname{span}\{\vec{u_1}, \vec{u_2}\} = W$. then $\{\vec{u_1}, \vec{u_2}\}$ is a basis of W.

Otherwise ... repeat this process and obtain larger and larger lin indep set of W.

Eventually will terminate and obtain $\beta = \{\vec{u_1}, \cdots, \vec{u_k}\}$ that spans W.

Then β is a basis of W and

$\dim W = k \leq n = \dim V$. since β is lin indep in V.

- If $\dim W = n$, then $\beta = \{\vec{u_1}, \cdots, \vec{u_n}\}$ is lin. indep in V
  and $|\beta| = n = \dim V$

  $\Rightarrow$ β is a basis of V.

  So $W = \text{Span}(\beta) = V$.

$\square$